

ONE DIMENSIONAL CONFORMAL METRIC FLOW II

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ABSTRACT. In this paper we continue our studies of the one dimensional conformal metric flows, which were introduced in [8]. In this part we mainly focus on evolution equations involving fourth order derivatives. The global existence and exponential convergence of metrics for the 1- Q and 4- Q flows are obtained.

1. INTRODUCTION

In [8] we initiated our study of one dimensional conformal curvature problem. Our research revealed the rich conformal structures on S^1 . It also has impacts in the study of affine geometry and its application to image processing.

Recall that if (S^1, g_s) is the unit circle with the induced metric $g_s = d\theta \otimes d\theta$ from \mathbf{R}^2 , for any metric g on S^1 (for example, this metric could be given by reparametrizing the circle), we write $g := d\sigma \otimes d\sigma = v^{-\frac{4}{3}}g_s$ for some positive function v . We then introduce a general α -scalar curvature of g for any positive constant α by

$${}^\alpha R_g = v(\alpha(v^{\frac{1}{3}})_{\theta\theta} + v^{\frac{1}{3}}).$$

The α -scalar curvature flows for $\alpha = 1$ and 4 were studied in [9], where the exponential convergence of metrics were obtained.

We further define a general α - Q curvature of g for any positive constant α by

$${}^\alpha Q_g = v^{\frac{5}{3}}\left(\frac{\alpha^2}{9}v_{\theta\theta\theta\theta} + \frac{10\alpha}{9}v_{\theta\theta} + v\right).$$

Thus ${}^\alpha Q_{g_s} = 1$. The corresponding α -conformal ${}^\alpha P_g$ operator of g is defined by

$${}^\alpha P_g f = \frac{\alpha^2}{9}\Delta_g^2 f + \frac{10\alpha}{9}\nabla_g({}^\alpha R_g \nabla_g f) + {}^\alpha Q_g f,$$

where $\nabla_g = D_\sigma$, $\Delta_g = D_{\sigma\sigma}$ and ${}^\alpha R_g$ is the α -scalar curvature of g .

We shall suppress the superscript “ α ” if no confusion would result. It is proved in [8] that P_g is a conformal covariant.

Proposition 1. *If $g_2 = \varphi^{-\frac{4}{3}}g_1$, then $Q_{g_2} = \varphi^{\frac{5}{3}}P_{g_1}\varphi$ and $P_{g_2}\psi = \varphi^{\frac{5}{3}}P_{g_1}(\psi\varphi)$, for any $\psi \in C^4(S^1)$.*

The general α - Q curvature flow is introduced as

$$(1.1) \quad \partial_t g = {}^\alpha Q_g g.$$

We will see in Section 2 that it is equivalent to the normalized α - Q curvature flow:

$$(1.2) \quad \partial_t g = ({}^\alpha Q_g - {}^\alpha \overline{Q}_g)g, \quad L(0) = 2\pi,$$

where ${}^\alpha \overline{Q}_g = \int {}^\alpha Q_g d\sigma / \int d\sigma$ and $L(t) = \int d\sigma$. It will be clear that this flow is in fact the gradient flow of total curvature ${}^\alpha \overline{Q}_g$ (see Lemma 2 below). We pointed out in [8] that two cases of $\alpha = 1$ and $\alpha = 4$ are of special interest. In this paper we

shall focus on these two cases. We will prove the global existence of the flows and exponential convergence of metrics for these two flows.

Recently there are some beautiful results on Q -curvature flow equations, though all of them focus on higher dimensional cases (for dimension $n \geq 4$). The global existence and convergence of higher order flow on general compact manifolds were obtained by Brendle under the condition of smaller total Q -curvature than that of the sphere with standard metric [3]. The convergence of Q -curvature flow on S^4 with the initial metric in the same conformal class of the standard metric was later obtained in [4]. The flow approach to the prescribing Q -curvature on S^4 is carried out by Malchiodi and Struwe [7]. There are two main ingredients in the proof of global existence and convergence of Q -curvature flow on S^4 . One is the sharp inequality involving higher order derivatives which guarantees the lower bound for a certain functional (see, for example, Branson, Chang and Yang [2], and Beckner [1]); The other is the new approach to the flow equations via integral estimates (see, for example, Chen [5], and Schwetlick and Struwe [11]).

Even though our flow is on one dimensional circle, we face the similar difficulty. For $\alpha = 4$, the extremal metric was classified in [8] (see, also Hang [6]). We thus can establish the global and convergence of 4- Q -curvature flow along the line as we just described.

Theorem 1. *There is a unique smooth solution $g(t)$, $t \in [0, \infty)$ to the flow equation (1.2) for any given initial metric $g_0 = v^{-\frac{4}{3}}(\theta, 0)g_s$ on S^1 . Moreover $g(t)$ converges exponentially to a smooth metric $g(\infty)$ and the 4- Q -curvature of $(S^1, g(\infty))$ is constant.*

The case of $\alpha = 1$ is more subtle. In [8], we proved the existence of sharp inequality (Theorem 3 in [8], see Remark 6 there for more comments), but were not able to classify the extremal metrics. To prove the exponential convergence of metrics under the 1- Q -curvature flow, one needs to classify all extremal metrics and to know the precise sharp constant. Now we can achieve this:

Theorem 2. *For $u(\theta) \in H^2(S^1)$ and $u > 0$, if u satisfies*

$$(1.3) \quad \int_0^{2\pi} \frac{\cos^3(\theta + \alpha)}{u^{5/3}(\theta)} d\theta = 0$$

for all $\alpha \in [0, 2\pi)$, then

$$\int_0^{2\pi} (u_{\theta\theta}^2 - 10u_{\theta}^2 + 9u^2) d\theta \left(\int_0^{2\pi} u^{-2/3}(\theta) d\theta \right)^3 \geq 144\pi^4.$$

More over, if u_0 is an extremal function, then the 1- Q -curvature of $u_0^{-4/3}g_s$ is a constant, and

$$(1.4) \quad u_0(\theta) = c \left(\lambda^2 \cos^2(\theta - \beta) + \lambda^{-2} \sin^2(\theta - \beta) \right)^{\frac{3}{2}},$$

for some $\lambda, c > 0$ and $\beta \in [0, 2\pi)$.

With this classification of extremal metrics, we are able to prove the exponential convergence of metrics under the 1- Q -curvature flow.

Theorem 3. *Suppose the initial metric $g_0 = v^{-\frac{4}{3}}(\theta, 0)g_s$ on S^1 satisfies the orthogonal condition*

$$\int_0^{2\pi} \frac{\cos^3(\theta + \beta)}{v^{\frac{5}{3}}} d\theta = 0, \quad \text{for any } \beta \in [0, 2\pi).$$

Then there is a unique solution $g(t)$ to the flow equation (1.2) for $t \in [0, \infty)$. Moreover $g(t)$ converges exponentially to a smooth metric $g(\infty)$ and the 1- Q -curvature of $(S^1, g(\infty))$ is constant.

The paper is organized as follows. In Section 2, we derive some basic properties about the flow and prove the global existence of the flow when $\alpha = 1$ or 4. The L^∞ convergences of the $\alpha - Q$ curvature for $\alpha = 1$ and 4 are obtained in Section 3. Using integral estimates, we then prove the exponential convergence of metric for 4- Q flow in Section 4, and complete the proof of Theorem 1. In Section 5, we first classify all the constant 1- Q curvature metrics on S^1 , thus complete the proof of Theorem 2; Then using a similar argument as in Section 4 we prove the exponential convergence of metric for 1- Q flow, thereby complete the proof of Theorem 3.

Throughout this paper, we use C, C_1, C_2, \dots , to represent some various positive constants.

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2. BASIC PROPERTIES AND GLOBAL EXISTENCE

In this section, we shall derive some basic equations for α -scalar curvature, α - Q curvature and conformal factor function under the flow. We then derive the *a priori* L^∞ estimates (depending on time) for metrics. From the estimates we obtain the global existence for the flows.

We first show that the flow equation (1.1) for $g(t)$ is equivalent to a normalized flow (1.2). In fact, if we choose

$$\hat{g}(t) = \frac{4\pi^2}{L^2(0)} \exp\left(-\int_0^t \overline{Q}_g(\tau) d\tau\right) g(t),$$

where $L(t) = \int d\sigma(t)$ and $g(t) = d\sigma(t) \otimes d\sigma(t)$, and a new time variable

$$\hat{t} = \frac{16\pi^4}{L^4(0)} \int_0^t \exp\left(-2 \int_0^\delta \overline{Q}_g(\tau) d\tau\right) d\delta,$$

then equation (1.1) can be written as

$$\partial_{\hat{t}} \hat{g} = \left(\hat{Q}_{\hat{g}} - \frac{\int \hat{Q}_{\hat{g}} d\hat{\sigma}}{\int d\hat{\sigma}} \right) \hat{g}.$$

From now on, we shall focus on the normalized flow (1.2).

Lemma 1. *Along flow equation (1.2) with $g(\sigma, t) = v^{-\frac{4}{3}}g_s$, curvatures $R = R_g$ and $Q = Q_g$ satisfy*

$$(2.1) \quad R_t = -\frac{\alpha}{4} \Delta Q - R(Q - \overline{Q})$$

and

$$(2.2) \quad Q_t = -\frac{\alpha^2}{12} \Delta^2 Q - \frac{5\alpha}{6} \nabla(R \nabla Q) - 2Q(Q - \overline{Q}),$$

respectively. Metric satisfies

$$(2.3) \quad \partial_t(d\sigma) = \frac{1}{2}(Q - \overline{Q})d\sigma,$$

and v satisfies

$$(2.4) \quad v_t = -\frac{3}{4}(Q - \overline{Q})v.$$

Proof.

$$(Q - \overline{Q})g = \partial_t g = \left(-\frac{4}{3}\right)v^{-\frac{7}{3}}v_t g_s = \left(-\frac{4}{3}\right)v^{-1}v_t g,$$

that is $v_t = -\frac{3}{4}(Q - \overline{Q})v$. Thus

$$\partial_t(d\sigma) = (v^{-\frac{2}{3}}d\theta)_t = -\frac{2}{3}v^{-\frac{5}{3}}v_t d\theta = \frac{1}{2}(Q - \overline{Q})d\sigma.$$

Using the conformal invariance of L_g (see, for example, [9]) and P_g , we have

$$\begin{aligned} R_t &= (vL_{g_s}(v^{\frac{1}{3}}))_t = v_t L_{g_s}(v^{\frac{1}{3}}) + vL_{g_s}\left(\frac{1}{3}v^{-\frac{2}{3}}v_t\right) \\ &= -\frac{3}{4}(Q - \overline{Q})R + \frac{1}{3}L_g\left(\frac{v_t}{v}\right) \\ &= -\frac{\alpha}{4}\Delta Q - R(Q - \overline{Q}), \end{aligned}$$

and

$$\begin{aligned} Q_t &= (v^{\frac{5}{3}}P_{g_s}(v))_t = \frac{5}{3}v^{\frac{2}{3}}v_t P_{g_s}(v) + v^{\frac{5}{3}}P_{g_s}\left(v\frac{v_t}{v}\right) \\ &= -\frac{5}{4}(Q - \overline{Q})Q + P_g\left(\frac{v_t}{v}\right) \\ &= -\frac{\alpha^2}{12}\Delta^2 Q - \frac{5\alpha}{6}\nabla(R\nabla_g Q) - 2Q(Q - \overline{Q}). \end{aligned}$$

□

It follows from (2.3) that

$$\partial_t \int_{S^1} d\sigma = \int_{S^1} \frac{1}{2}(Q_g - \overline{Q}_g)d\sigma = 0.$$

Thus flow (1.2) preserves the arc length with respect to metric g (i.e. $\int_0^{2\pi} v^{-\frac{2}{3}}d\theta = L(0) = 2\pi$). Moreover, along the flow, we see from the following lemma that the total Q curvature is strictly decreasing unless Q_g is a constant.

Lemma 2. *Along flow (1.2), we have*

$$(2.5) \quad \partial_t \overline{Q}_g = -\frac{3}{4\pi} \int_{S^1} (Q_g - \overline{Q}_g)^2 d\sigma.$$

Proof.

$$\begin{aligned}
\partial_t \overline{Q}_g &= \frac{1}{2\pi} \int_{S^1} (Q_g)_t d\sigma + \frac{1}{2\pi} \int_{S^1} Q_g \partial_t(d\sigma) \\
&= \frac{1}{2\pi} \int_{S^1} (-2Q_g)(Q_g - \overline{Q}_g) d\sigma + \frac{1}{4\pi} \int_{S^1} Q_g(Q_g - \overline{Q}_g) d\sigma \\
&= -\frac{3}{4\pi} \int_{S^1} Q_g(Q_g - \overline{Q}_g) d\sigma = -\frac{3}{4\pi} \int_{S^1} (Q_g - \overline{Q}_g)^2 d\sigma \leq 0.
\end{aligned}$$

□

Note that (2.4) can also be written as

$$(2.6) \quad v_t = -\frac{\alpha^2}{12} v^{\frac{8}{3}} \Delta_{g_s}^2 v - \frac{5\alpha}{6} v^{\frac{8}{3}} \Delta_{g_s} v - \frac{3}{4} v^{\frac{11}{3}} + \frac{3}{4} \overline{Q}_g v.$$

We are now ready to prove the global existence for $\alpha = 4$ and $\alpha = 1$.

Proposition 2. *Suppose that $g(t) = v^{-\frac{4}{3}} g_s$ satisfies the flow equation (1.2) on $[0, T)$ for $\alpha = 4$. Then there exists $C = C(T)$, such that*

$$\frac{1}{C} < v(t) < C \quad \text{and} \quad \|v(t)\|_{H^2} < C \quad \text{on } [0, T).$$

Proof. For any given $\lambda > 0$ and $\beta \in [0, 2\pi)$, let

$$\Psi_{\lambda, \beta}(\theta) = \left(\lambda^2 \cos^2 \frac{\theta - \beta}{2} + \lambda^{-2} \sin^2 \frac{\theta - \beta}{2} \right)^{3/2},$$

$$\omega_{\lambda, \beta}(\theta) = \beta + \int_{\beta}^{\theta} \Psi_{\lambda, \beta}^{-\frac{2}{3}} d\theta = \begin{cases} \beta + 2 \arctan(\lambda^{-2} \tan \frac{\theta - \beta}{2}), & \text{if } 0 \leq \theta - \beta \leq \pi \\ \beta + 2 \arctan(\lambda^{-2} \tan \frac{\theta - \beta}{2}) + 2\pi, & \text{if } \pi < \theta - \beta \leq 2\pi \end{cases}$$

and

$$(\mathcal{T}_{\lambda, \beta} v)(\theta) = v(\omega_{\lambda, \beta}(\theta)) \Psi_{\lambda, \beta}(\theta).$$

By Lemma 3 in [8], we know for any given $t \in [0, T)$, there exist $\lambda = \lambda(t) > 0, \beta \in [0, 2\pi)$ such that $u(\theta, t) := (\mathcal{T}_{\lambda, \beta} v)(\theta)$ satisfies $\int_0^{2\pi} u(\theta, t) \cos \theta d\theta = \int_0^{2\pi} u(\theta, t) \sin \theta d\theta = 0$. For any positive function $\varphi \in H^2(S^1)$ define functional

$$F(\varphi) = \frac{16}{9} \left(\int_0^{2\pi} (\varphi_{\theta\theta}^2 - \frac{5}{2} \varphi_{\theta}^2 + \frac{9}{16} \varphi^2) d\theta \right) \left(\int_0^{2\pi} \varphi^{-\frac{2}{3}} d\theta \right)^3.$$

Suppose the Fourier expansion of $u(\theta, t)$ is

$$u(\theta, t) = a_0 + \sum_{k=2}^{\infty} a_k \cos(k\theta - \gamma_k).$$

Then we have (noting that $\int_0^{2\pi} u^{-2/3} d\theta = 2\pi$)

$$F(u) = (2\pi)^4 \left(a_0^2 + \frac{8}{9} \sum_{k=2}^{\infty} (k^4 - \frac{5}{2} k^2 + \frac{9}{16}) a_k^2 \right),$$

which implies

$$F(u) \geq C \int_0^{2\pi} (u_{\theta\theta}^2 + u^2) d\theta$$

for some constant $C > 0$. On the other hand, from the conformal covariance of F , we have $F(u(\theta, t)) = F(v(\theta, t)) = (2\pi)^4 \overline{Q}(t) \leq (2\pi)^4 \overline{Q}(0)$. It follows that $\|u(t)\|_{H^2}$

and therefore $\|u(t)\|_{C^{1,a}}$ ($a \in (0, \frac{1}{2}]$) is bounded on $[0, T]$. Since $\int_0^{2\pi} u^{-\frac{2}{3}} d\theta = 2\pi$ we know that there exists a constant C_1 not depending on T , such that $\frac{1}{C_1} \leq u \leq C_1$ on $[0, 2\pi] \times [0, T]$.

In order to obtain the estimates on $v(\theta, t)$, it suffices to prove that $\lambda(t)$ is bounded on $[0, T]$. Suppose not, there exists a sequence $t_i \rightarrow T$, such that $\lambda(t_i) \rightarrow \infty$. Without loss of generality, we may assume that $\beta(t_i) \rightarrow 0$. Then for any $\epsilon > 0$,

$$\lim_{i \rightarrow \infty} \left(\int_{-\epsilon}^{\epsilon} v(t_i)^{-\frac{2}{3}} d\theta + \int_{\pi-\epsilon}^{\pi+\epsilon} v(t_i)^{-\frac{2}{3}} d\theta \right) = 2\pi.$$

On the other hand, for any $t > \tilde{t}$ we have

$$\begin{aligned} \left| \int_{-\epsilon}^{\epsilon} v(t)^{-\frac{2}{3}} d\theta - \int_{-\epsilon}^{\epsilon} v(\tilde{t})^{-\frac{2}{3}} d\theta \right| &= \left| \int_{\tilde{t}}^t \left(\int_{-\epsilon}^{\epsilon} v(t)^{-\frac{2}{3}} d\theta \right)_t dt \right| \\ &\leq \int_{\tilde{t}}^t \int_0^{2\pi} \frac{|Q - \overline{Q}|}{2} d\sigma dt \\ &\leq C_2(t - \tilde{t})^{\frac{1}{2}} \left(\int_{\tilde{t}}^t \int_0^{2\pi} (Q - \overline{Q})^2 d\sigma dt \right)^{\frac{1}{2}}. \end{aligned}$$

For fixed $t \in [0, T]$,

$$2\pi = \lim_{i \rightarrow \infty} \left(\int_{-\epsilon}^{\epsilon} + \int_{\pi-\epsilon}^{\pi+\epsilon} \right) v^{-\frac{2}{3}}(t_i) d\theta \leq \left(\int_{-\epsilon}^{\epsilon} + \int_{\pi-\epsilon}^{\pi+\epsilon} \right) v^{-\frac{2}{3}}(t) d\theta + C_3(T - t)^{\frac{1}{2}}.$$

Choosing t closed enough to T and choosing ϵ small enough we get a contradiction. \square

For $\alpha = 1$ we have a similar proposition:

Proposition 3. *Suppose that $g(t) = v^{-\frac{4}{3}} g_s$ satisfies the flow equation (1.2) on $[0, T]$ for $\alpha = 1$. Then there exists $C = C(T)$, such that*

$$\frac{1}{C} < v(t) < C \quad \text{and} \quad \|v(t)\|_{H^2} < C \quad \text{on } [0, T].$$

Proof. For any $\lambda > 0$, let

$$\Gamma_\lambda(\theta) = (\lambda^2 \cos^2 \theta + \lambda^{-2} \sin^2 \theta)^{3/2}, \quad \sigma_\lambda(\theta) = \int_0^\theta \Gamma_\lambda^{-2/3} d\theta,$$

and

$$(\mathbf{T}_\lambda u)(\theta) := u(\sigma_\lambda(\theta)) \Gamma_\lambda(\theta).$$

Then again by Lemma 3 in [8] we know that for any given $t \in [0, T]$, there exist $\lambda = \lambda(t) > 0, \beta \in [0, 2\pi)$ such that $u(\theta, t) := (\mathbf{T}_{\lambda, \beta} v)(\theta)$ satisfies $\int_0^{2\pi} u(t) \cos 2\theta d\theta = \int_0^{2\pi} u(t) \sin 2\theta d\theta = 0$. For any positive function $\varphi \in H^2(S^1)$ define functional

$$\mathbf{F}(\varphi) = \frac{1}{9} \left(\int_0^{2\pi} (\varphi_{\theta\theta}^2 - 10\varphi_\theta^2 + 9\varphi^2) d\theta \right) \left(\int_0^{2\pi} \varphi^{-\frac{2}{3}} d\theta \right)^3.$$

Suppose the Fourier expansion of $u(\theta, t)$ is

$$u(\theta, t) = a_0 + a_1 \cos(\theta - \gamma_1) + \sum_{k=3}^{\infty} a_k \cos(k\theta - \gamma_k).$$

Then we have

$$\mathbf{F}(u) = (2\pi)^4 \left(a_0^2 + \frac{1}{18} \sum_{k=4}^{\infty} (k^4 - 10k^2 + 9) a_k^2 \right).$$

Integrating the nonnegative function $u(\theta)(1 \pm \cos(\theta - \gamma_1))$ we have

$$0 \leq \int_0^{2\pi} u(\theta)(1 \pm \cos(\theta - \gamma_1)) d\theta = 2\pi a_0 \pm \pi a_1,$$

which implies $|a_1| \leq 2a_0$. Similarly, integrating the nonnegative function $u(\theta)(1 \pm \cos(3\theta - \gamma_1))$ we have $|a_3| \leq 2a_0$. Hence

$$\mathbf{F}(u) \geq C \|u\|_{H^2}^2$$

for some constant $C > 0$. Also from the conformal covariance of \mathbf{F} , we have $\mathbf{F}(u(\theta, t)) = \mathbf{F}(v(\theta, t)) = (2\pi)^4 \overline{Q}(t) \leq (2\pi)^4 \overline{Q}(0)$. It follows that $\|u(t)\|_{H^2}$ is bounded on $[0, T)$. The rest of the proof will be similar to the proof of Proposition 2. \square

Using a similar argument in [3] we obtain the estimates on higher derivative of $v(\theta, t)$ as follows.

Proposition 4. *Suppose that $g(t) = v^{-\frac{4}{3}} g_s$ satisfies the flow equation (1.2) on $[0, T)$ for $\alpha = 1$ or 4. Then $\|v(t)\|_{H^{2k}(S^1)}$ is bounded on $[0, T)$ for any $k \in \mathbf{N}$.*

Proof.

$$\partial_t \int_0^{2\pi} (v^{(2k)})^2 d\theta = 2 \int_0^{2\pi} v^{(2k)} (v_t)^{(2k)} d\theta = -\frac{3}{2} \int_0^{2\pi} v^{(2k)} ((Q - \overline{Q})v)^{(2k)} d\theta.$$

Since $Q = \frac{1}{9} v^{\frac{5}{3}} (\alpha^2 v^{(4)} + 10\alpha v^{(2)} + 9v)$, we obtain from Proposition 2 and 3 that

$$\begin{aligned} \partial_t \int_0^{2\pi} (v^{(2k)})^2 d\theta &\leq -\frac{\alpha^2}{6} \int_0^{2\pi} (v^{(2k+2)})^2 v^{\frac{8}{3}} d\theta + C_1 \sum_{k_1, \dots, k_m} \int_0^{2\pi} \prod_{i=1}^m |v^{(k_i)}| d\theta \\ &\leq -C_2 \int_0^{2\pi} (v^{(2k+2)})^2 v^{\frac{8}{3}} d\theta + C_1 \sum_{k_1, \dots, k_m} \int_0^{2\pi} \prod_{i=1}^m |v^{(k_i)}| d\theta, \end{aligned}$$

where Σ is taken over all m -tuples k_1, \dots, k_m with $m \geq 3$, which satisfy $1 \leq k_i \leq 2k+1$ and $k_1 + \dots + k_m \leq 4k+4$.

For each m -tuple k_1, \dots, k_m , let $r_i = \max\{0, \frac{k_i - \frac{1}{m} - \frac{3}{2}}{2k}\}$. Then we have $r := \theta_1 + \dots + \theta_m < 2$ and $\|v^{(k_i)}\|_{L^m} \leq C \|v\|_{H^{k_i - \frac{1}{m} + \frac{1}{2}}} \leq C \|v\|_{H^2}^{1-r_i} \|v\|_{H^{2k+2}}^{r_i}$. It follows that

$$\begin{aligned} \partial_t \int_0^{2\pi} (v^{(2k)})^2 d\theta &\leq -C_1 \int_0^{2\pi} (v^{(2k+2)})^2 d\theta + C_2 \sum_{k_1, \dots, k_m} \int_0^{2\pi} \prod_{i=1}^m |v^{(k_i)}| d\theta \\ &\leq -C_1 \int_0^{2\pi} (v^{(2k+2)})^2 d\theta + C_3 \sum_{k_1, \dots, k_m} \prod_{i=1}^m \|v\|_{H^{k_i - \frac{1}{m} + \frac{1}{2}}} \\ &\leq -C_1 \int_0^{2\pi} (v^{(2k+2)})^2 d\theta + C_4 \sum_{k_1, \dots, k_m} \|v\|_{H^2}^{m-r} \|v\|_{H^{2k+2}}^r \\ &\leq -C_1 \int_0^{2\pi} (v^{(2k+2)})^2 d\theta + C_5 \|v\|_{H^{2k+2}}^r \leq C_6. \end{aligned}$$

Hence $\int_0^{2\pi} (v^{(2k)})^2 d\theta$ is bounded on $[0, T]$. \square

For $\alpha = 4$ we know that ${}^\alpha P_g$ is positive. From the above proposition, we immediately get that the 4- Q -flow exists on $[0, \infty)$. To show the global existence of the flow for $\alpha = 1$, we need another lemma.

Lemma 3. *Suppose that $g(t) = v^{-\frac{4}{3}} g_s$ satisfies the flow equation (1.2) on $[0, T]$ for $\alpha = 1$. If $\int_0^{2\pi} \cos^3(\theta + \alpha) \cdot v^{-5/3}(\theta, 0) d\theta = 0$ for all $\alpha \in [0, 2\pi)$, then for all $t > 0$,*

$$(2.7) \quad \int_0^{2\pi} \cos^3(\theta + \alpha) \cdot v^{-5/3}(\theta, t) d\theta = 0$$

for all $\alpha \in [0, 2\pi)$.

Proof. From (2.4) and the definition of 1- Q curvature we have

$$\begin{aligned} \partial_t \int_0^{2\pi} \cos^3(\theta + \alpha) \cdot v^{-5/3}(\theta, t) d\theta &= -\frac{5}{3} \int_0^{2\pi} \cos^3(\theta + \alpha) \cdot v^{-8/3}(\theta, t) v_t(\theta, t) d\theta \\ &= \frac{5}{4} \int_0^{2\pi} {}^1 Q \cos^3(\theta + \alpha) \cdot v^{-5/3}(\theta, t) d\theta - \frac{5 {}^1 \overline{Q}}{4} \int_0^{2\pi} \cos^3(\theta + \alpha) \cdot v^{-5/3}(\theta, t) d\theta \\ &= -\frac{5 {}^1 \overline{Q}}{4} \int_0^{2\pi} \cos^3(\theta + \alpha) \cdot v^{-5/3}(\theta, t) d\theta. \end{aligned}$$

Thus

$$\int_0^{2\pi} \cos^3(\theta + \alpha) \cdot v^{-5/3}(\theta, t) d\theta = C e^{-\int_0^t {}^1 \overline{Q}(\tau) d\tau}.$$

Since $\int_0^{2\pi} \cos^3(\theta + \alpha) \cdot v^{-5/3}(\theta, 0) d\theta = 0$, we have $C = 0$, thus $\int_0^{2\pi} \cos^3(\theta + \alpha) \cdot v^{-5/3}(\theta, t) d\theta = 0$. \square

For $\alpha = 1$ we know ${}^\alpha P_g$ is positive on

$$\{u \in H^4(S^1) : u > 0, \int_0^{2\pi} \cos^3(\theta + \alpha) \cdot u^{-5/3}(\theta, t) d\theta = 0, \forall \alpha \in [0, 2\pi)\},$$

see, for example, Theorem 3 in [8]. The global existence of 1- Q -curvature flow then follows from Proposition 4 and Lemma 3.

3. L^∞ CONVERGENCE OF ${}^\alpha Q$ ALONG α - Q -FLOW

In this section, we shall follow [9] closely to derive the L^∞ norm convergence for curvatures. Throughout the rest of the paper, we will only consider $\alpha = 1$ or $\alpha = 4$; In the case of $\alpha = 1$, we always assume the initial metric satisfies the orthogonal condition (2.7) (thus always satisfies (2.7) along the flow by Lemma 3); we also denote $L^p = L^p(d\sigma)$.

For $p \geq 2$, we define

$$G_p(t) := \int_0^{2\pi} |Q - \overline{Q}|^p d\sigma.$$

By (2.5) and (2.2) we have

$$(3.1) \quad \partial_t G_2 = 2 \int_0^{2\pi} (Q - \overline{Q}) Q_t d\sigma + \frac{1}{2} \int_0^{2\pi} (Q - \overline{Q})^3 d\sigma,$$

and

$$(3.2) \quad \int_0^{2\pi} (Q - \overline{Q}) Q_t d\sigma = -\frac{\alpha^2}{12} \|Q_{\sigma\sigma}\|_{L^2}^2 + \frac{5\alpha}{6} \int_0^{2\pi} R Q_\sigma^2 d\sigma - 2 \int_0^{2\pi} Q(Q - \overline{Q})^2 d\sigma.$$

We have the following well-known interpolation inequality.

Lemma 4.

$$\int_0^{2\pi} Q_\sigma^4 d\sigma \leq C \|Q - \overline{Q}\|_{L^2} \|Q_{\sigma\sigma}\|_{L^2}^3.$$

Using the above lemma and Young's inequality we obtain that for any $a > 0$:

$$(3.3) \quad \left| \int_0^{2\pi} R Q_\sigma^2 d\sigma \right| \leq \|R\|_{L^2} \|Q_\sigma\|_{L^4}^2 \leq C \|R\|_{L^2} \|Q - \overline{Q}\|_{L^2}^{\frac{1}{2}} \|Q_{\sigma\sigma}\|_{L^2}^{\frac{3}{2}} \\ \leq C \|R\|_{L^2} \left(\frac{1}{4a^4} \|Q - \overline{Q}\|_{L^2}^2 + \frac{3}{4} a^{\frac{4}{3}} \|Q_{\sigma\sigma}\|_{L^2}^2 \right).$$

Since $\|R\|_{L^2}^2 = 2\pi \overline{Q}$ is bounded (see Remark 7 of [8]), it follows from (3.1), (3.2) and (3.3) that

$$\partial_t G_2 \leq C_1 G_2 - \frac{7}{2} \int_0^{2\pi} (Q - \overline{Q})^3 d\sigma - C_2 \int_0^{2\pi} Q_{\sigma\sigma}^2 d\sigma.$$

Noticing that

$$\left| \int_0^{2\pi} (Q - \overline{Q})^3 d\sigma \right| \leq \|Q - \overline{Q}\|_{L^2}^{\frac{3}{2}} \|Q - \overline{Q}\|_{L^6}^{\frac{3}{2}} \leq \frac{1}{4a^4} \|Q - \overline{Q}\|_{L^2}^6 + \frac{3}{4} a^{\frac{4}{3}} \|Q - \overline{Q}\|_{L^6}^2$$

and $\|Q - \overline{Q}\|_{L^6}^2 \leq C_3 \|Q_{\sigma\sigma}\|_{L^2}^2$, we obtain that

$$(3.4) \quad \partial_t G_2 \leq C_4 (G_2 + G_2^3) - C_5 \|Q_{\sigma\sigma}\|_{L^2}^2.$$

Lemma 5.

$$\lim_{t \rightarrow \infty} G_2(t) = 0, \quad \int_0^\infty \int_0^{2\pi} Q_{\sigma\sigma}^2 d\sigma dt < \infty.$$

Proof. From Lemma 2 we know that $\int_0^\infty G_2(t) dt < \infty$. Therefore for any $\epsilon > 0$, there exists $t_\epsilon > 0$, such that $G_2(t_\epsilon) < \epsilon$ and $\int_{t_\epsilon}^\infty G_2(t) dt < \epsilon$. If $\epsilon < 1/(1 + 2C_4)$, we must have that $G_2(t) \leq 1$ for all $t > t_\epsilon$. In fact, if not, let $t_* > t_\epsilon$ be the first time such that $G_2(t_*) = 1$. Integrating (3.4) from t_ϵ to t_* , we obtain that

$$1 - G_2(t_\epsilon) \leq C_4(\epsilon + \epsilon),$$

which implies $\epsilon \geq 1/(1 + 2C_4)$. Contradiction. For $\epsilon < 1/(1 + 2C_4)$ and $t > t_\epsilon$, integration (3.4) from t_ϵ to t , we obtain that

$$G_2(t) \leq G_2(t_\epsilon) + C_4(\epsilon + \epsilon) \leq \epsilon + 2C_4\epsilon.$$

Hence $\lim_{t \rightarrow \infty} G_2(t) = 0$ and $\int_0^\infty (\int_0^{2\pi} Q_{\sigma\sigma}^2 d\sigma) dt < \infty$. \square

Direct computation yields

$$(3.5) \quad \partial_t \|Q_\sigma\|_{L^2}^2 = -\frac{\alpha^2}{6} \|Q_{\sigma\sigma\sigma}\|_{L^2}^2 - \frac{5\alpha}{3} \int_0^{2\pi} Q_\sigma Q_{\sigma\sigma\sigma} R d\sigma \\ - 4\overline{Q} \|Q_\sigma\|_{L^2}^2 - \frac{17}{2} \int_0^{2\pi} Q_\sigma^2 (Q - \overline{Q}) d\sigma.$$

For any $t \geq 0$, choose $\sigma_0 > 0$ such that $Q_\sigma(\sigma_0) = 0$. Then

$$|Q_\sigma| = |Q_\sigma - Q_\sigma(\sigma_0)| \leq \|Q_{\sigma\sigma}\|_{L^1} \leq \sqrt{2\pi} \|Q_{\sigma\sigma}\|_{L^2},$$

which implies $\|Q_\sigma\|_{L^\infty} \leq \sqrt{2\pi}\|Q_{\sigma\sigma}\|_{L^2}$. It follows that

$$\begin{aligned}
 (3.6) \quad \left| \int_0^{2\pi} Q_\sigma Q_{\sigma\sigma\sigma} R d\sigma \right| &\leq \frac{1}{C} \|Q_{\sigma\sigma\sigma}\|_{L^2}^2 + C \int_0^{2\pi} Q_\sigma^2 R^2 d\sigma \\
 &\leq \frac{1}{C} \|Q_{\sigma\sigma\sigma}\|_{L^2}^2 + C \|Q_\sigma\|_{L^\infty}^2 \|R\|_{L^2}^2 \\
 &\leq \frac{1}{C} \|Q_{\sigma\sigma\sigma}\|_{L^2}^2 + 4\pi^2 C \|Q_{\sigma\sigma}\|_{L^2}^2 \overline{Q},
 \end{aligned}$$

and

$$(3.7) \quad \left| \int_0^{2\pi} Q_\sigma^2 (Q - \overline{Q}) d\sigma \right| \leq \|Q_\sigma\|_{L^\infty}^2 \|Q - \overline{Q}\|_{L^1} \leq 2\pi \sqrt{2\pi} \|Q_{\sigma\sigma}\|_{L^2}^2 G_2^{\frac{1}{2}}.$$

Substituting (3.6) and (3.7) in (3.5) and noticing that $G_2(t) \rightarrow 0$, we obtain that

$$(3.8) \quad \partial_t \|Q_\sigma\|_{L^2}^2 \leq -C_1 \|Q_{\sigma\sigma\sigma}\|_{L^2}^2 + C_2 \|Q_{\sigma\sigma}\|_{L^2}^2 \leq C_2 \|Q_{\sigma\sigma}\|_{L^2}^2.$$

It follows from Lemma 5 that for any $\epsilon > 0$, there exists $t_\epsilon > 0$, such that

$$\|Q_{\sigma\sigma}\|_{L^2}^2(t_\epsilon) < \epsilon \quad \text{and} \quad \int_{t_\epsilon}^\infty \|Q_{\sigma\sigma}\|_{L^2}^2 dt < \epsilon.$$

For any $t > t_\epsilon$, integrating (3.8) from t_ϵ to t , we obtain that

$$\|Q_\sigma\|_{L^2}^2(t) \leq \|Q_\sigma\|_{L^2}^2(t_\epsilon) + C_2 \epsilon \leq (2\pi)^2 \|Q_{\sigma\sigma}\|_{L^2}^2(t_\epsilon) + C_2 \epsilon \leq C_3 \epsilon.$$

Hence $\|Q_\sigma\|_{L^2} \rightarrow 0$ as $t \rightarrow +\infty$, which implies

$$\lim_{t \rightarrow \infty} \|Q - \overline{Q}\|_{L^\infty} = 0.$$

4. EXPONENTIAL CONVERGENCE OF THE 4- Q -FLOW

We are now ready to derive the exponential convergence for the metrics under 4- Q -curvature flow and thus complete the proof of Theorem 1.

Suppose $g(t) = v^{-\frac{4}{3}}(\theta, t)g_s$ is a solution to the flow equation (1.2) for $\alpha = 4$. As in the proof of Proposition 2, for any $t \in [0, \infty)$, we can choose $\lambda = \lambda(t) > 0, \beta = \beta(t) \in [0, 2\pi)$ so that $u(\theta, t) := (\mathcal{T}_{\lambda, \beta} v)(\theta)$ satisfies

$$(4.1) \quad \int_0^{2\pi} u(t) \cos \theta d\theta = \int_0^{2\pi} u(t) \sin \theta d\theta = 0.$$

Then $u(\theta, t)$ is uniformly bounded in $H^2(S^1)$ for $t \in [0, \infty)$. Therefore there exists a sequence $t_n \rightarrow \infty$, such that $u(\theta, t_n) \rightharpoonup u_\infty(\theta)$ in $H^2(S^1)$. From Sobolev embedding theorem we have $u(\theta, t_n) \rightarrow u_\infty(\theta)$ in $C^{1,a}$ for any $a \in (0, \frac{1}{2})$ and $u \in C^{1,1/2}$. Since

$$\int_0^{2\pi} u^{-\frac{2}{3}}(\theta, t) d\theta = \int_0^{2\pi} v^{-\frac{2}{3}}(\theta, t) d\theta = 2\pi,$$

we obtain that $u_\infty(\theta) > 0$ and u_∞ satisfies

$$u_\infty^{\frac{5}{3}} \left(\frac{16}{9} (u_\infty)_{\theta\theta\theta\theta} + \frac{40}{9} (u_\infty)_{\theta\theta} + u_\infty \right) = Q_\infty,$$

where $Q_\infty = \lim_{t \rightarrow \infty} \overline{Q}$. It follows from (4.1) and the classification of solutions of the above ODE (see the proof of Theorem 4 in [8]) that $u_\infty = 1$. Using the same argument we can prove that any convergent subsequence of $u(\theta, t)$ converges to 1. Since $u(\theta, t)$ is uniformly bounded in $H^2(S^1)$, we have $\lim_{t \rightarrow \infty} u(\theta, t) = 1$. Hence $Q_\infty = 1$.

Define variable γ as the inverse of θ under map $\omega_{\lambda(t),\beta(t)}$, that is $\omega_{\lambda,\beta}(\gamma) = \theta$. Noting that

$$u(\theta, t) = v(\omega_{\lambda,\beta}(\theta))\Psi_{\lambda,\beta}(\theta), \quad \Psi_{\lambda,\beta}(\theta) = \left(\lambda^2 \cos^2 \frac{\theta - \beta}{2} + \lambda^{-2} \sin^2 \frac{\theta - \beta}{2} \right)^{3/2}$$

and $\omega_{\lambda,\beta}(\theta) = \beta + \int_{\beta}^{\theta} \Psi_{\lambda,\beta}^{-\frac{2}{3}} d\theta$ we have $d\sigma = v(\theta)^{-\frac{2}{3}} d\theta = u(\gamma)^{-\frac{2}{3}} d\gamma$. Since the 4-curvature and 4- Q curvature of metric $d\gamma \otimes d\gamma$ is 1, we obtain that

$$\begin{aligned} R(\theta(\gamma)) &= u(\gamma)(4(u^{\frac{1}{3}})'(\gamma) + u^{\frac{1}{3}}(\gamma)), \\ Q(\theta(\gamma)) &= u^{\frac{5}{3}}(\gamma)\left(\frac{16}{9}u_{\gamma\gamma\gamma}(\gamma) + \frac{40}{9}u_{\gamma\gamma}(\gamma) + u(\gamma)\right). \end{aligned}$$

It follows that $\lim_{t \rightarrow \infty} R(\theta) = 1$. Therefore

$$\begin{aligned} \partial_t G_2 &= -\frac{8}{3} \|Q_{\sigma\sigma}\|_{L^2}^2 + \frac{20}{3} \int_0^{2\pi} R Q_{\sigma}^2 d\sigma - 4\bar{Q} \|Q - \bar{Q}\|_{L^2}^2 - \frac{7}{2} \int_0^{2\pi} (Q - \bar{Q})^3 d\sigma \\ (4.2) \quad &= -\frac{8}{3} \|Q_{\sigma\sigma}\|_{L^2}^2 + \left(\frac{20}{3} + o(1)\right) \|Q_{\sigma}\|_{L^2}^2 - (4 - o(1)) \|Q - \bar{Q}\|_{L^2}^2, \end{aligned}$$

where $o(1) \rightarrow 0$ as $t \rightarrow +\infty$.

Consider the Fourier series of Q :

$$(4.3) \quad Q = \bar{Q} + \sum_{n=1}^{\infty} (a_n \cos(n\sigma) + b_n \sin(n\sigma)) = \tilde{c} + \sum_{n=1}^{\infty} (\tilde{a}_n \cos(n\gamma) + \tilde{b}_n \sin(n\gamma)).$$

Since $u(t) \rightarrow 1$ as $t \rightarrow \infty$, we obtain that

$$(4.4) \quad a_n = \tilde{a}_n + o(1)G_2^{\frac{1}{2}}, \quad b_n = \tilde{b}_n + o(1)G_2^{\frac{1}{2}}, \quad n = 0, 1, 2, 3, \dots,$$

where $a_0 = \bar{Q}$ and $\tilde{a}_0 = \tilde{c}$. As in [9] we need estimates on a_1 and b_1 .

Lemma 6. *For ϕ smooth we have*

$$\int_0^{2\pi} (16\phi^{(4)}(\theta) + 40\phi''(\theta) + 9\phi(\theta)) \left(\frac{2}{3}\phi'(\theta) \cos \theta + \phi(\theta) \sin \theta\right) d\theta = 0.$$

Proof. Integrating by parts we have

$$(4.5) \quad \int_0^{2\pi} \phi'' \phi''' \cos \theta d\theta = \frac{1}{2} \int_0^{2\pi} (\phi'')^2 \sin \theta d\theta.$$

It follows that

$$\begin{aligned} & \int_0^{2\pi} (16\phi^{(4)} + 40\phi'') \left(\frac{2}{3}\phi' \cos \theta + \phi \sin \theta\right) d\theta \\ &= \int_0^{2\pi} \phi'' \left(\frac{32}{3}\phi''' \cos \theta - \frac{16}{3}\phi'' \sin \theta + 48\phi' \cos \theta + 24\phi \sin \theta\right) d\theta \\ &= 0 + \int_0^{2\pi} (48\phi'' \phi' \cos \theta + 24\phi'' \phi \sin \theta) d\theta \\ &= 24 \int_0^{2\pi} (\phi')^2 \sin \theta d\theta - 24 \int_0^{2\pi} (\phi')^2 \sin \theta d\theta - 12 \int_0^{2\pi} \phi^2 \sin \theta d\theta \\ &= -12 \int_0^{2\pi} \phi^2 \sin \theta d\theta, \end{aligned}$$

where we used (4.5). Also

$$\int_0^{2\pi} 9\phi \cdot \left(\frac{2}{3}\phi' \cos \theta + \phi \sin \theta\right) d\theta = 12 \int_0^{2\pi} \phi^2 \sin \theta d\theta.$$

and the lemma follows. \square

The above lemma yields a Kazdan-Warner type identity:

Corollary 1. *Given $g = v^{-\frac{4}{3}}g_s$ with v smooth. Then 4- Q curvature of g satisfies*

$$(4.6) \quad \int_0^{2\pi} Q_\theta v^{-\frac{2}{3}} \cos \theta d\theta = \int_0^{2\pi} Q_\theta v^{-\frac{2}{3}} \sin \theta d\theta = 0.$$

Proof.

$$\begin{aligned} \int_0^{2\pi} Q_\theta v^{-\frac{2}{3}} \cos \theta d\theta &= - \int_0^{2\pi} Q \left(-\frac{2}{3} v^{-\frac{5}{3}} v_\theta \cos \theta - v^{-\frac{2}{3}} \sin \theta \right) d\theta \\ &= \frac{1}{9} \int_0^{2\pi} (16v^{(4)} + 40v'' + 9v) \left(\frac{2}{3} v' \cos \theta + v \sin \theta \right) d\theta = 0. \end{aligned}$$

Applying the above lemma to $\phi(\theta) = v(\theta + \frac{\pi}{2})$, we obtain that

$$\int_0^{2\pi} Q_\theta v^{-\frac{2}{3}} \sin \theta d\theta = 0.$$

\square

Using (4.6) and $\lim_{t \rightarrow \infty} u = 1$ we have the following computation

$$\begin{aligned} \tilde{a}_1 &= \frac{1}{\pi} \int_0^{2\pi} Q \cos \gamma d\gamma = -\frac{1}{\pi} \int_0^{2\pi} Q_\gamma \sin \gamma d\gamma \\ &= -\frac{1}{\pi} \int_0^{2\pi} Q_\gamma \sin \gamma (u^{-\frac{2}{3}}(\gamma) - u^{-\frac{2}{3}}(\gamma) + 1) d\gamma \\ &= -\frac{1}{\pi} \int_0^{2\pi} Q_\gamma \sin \gamma (-u^{-\frac{2}{3}}(\gamma) + 1) d\gamma + 0 \\ &= o(1) \left(\int_0^{2\pi} Q_\gamma^2 d\gamma \right)^{\frac{1}{2}} = o(1) \|Q_\sigma\|_{L^2(d\sigma)}. \end{aligned}$$

Similarly $\tilde{b}_1 = o(1) \|Q_\sigma\|_{L^2}$. It follows from (4.4) that $a_1, b_1 = o(1) \|Q_\sigma\|_{L^2}$.

From Fourier expansion (4.3) of Q we obtain that

$$\begin{aligned} \|Q_{\sigma\sigma}\|_{L^2}^2 &= \pi \sum_{n=1}^{\infty} n^4 (a_n^2 + b_n^2), \quad \|Q_\sigma\|_{L^2}^2 = \pi \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2), \\ G_2 &= \|Q - \overline{Q}\|_{L^2}^2 = \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2). \end{aligned}$$

It follows from (4.2) that

$$\partial_t G_2 = \pi \sum_{n=2}^{\infty} \left(-\frac{8}{3} n^4 + \frac{20}{3} n^2 - 4 \right) (a_n^2 + b_n^2) + o(\|Q_\sigma\|_{L^2}).$$

Hence there exists $a > 0$, such that $\partial G_2 \leq -a G_2$, which implies that

$$G_2(t) \leq C e^{-at}, \text{ for some } C > 0.$$

For any $T > 0$ and $\delta \in [0, 1]$, integrating (3.4) from T to $T + \delta$ and using the above inequality we have

$$\int_T^{T+\delta} \|Q_{\sigma\sigma}\|_{L^2}^2 dt \leq C_1 e^{-aT},$$

which implies

$$\int_T^{T+\delta} \|Q - \overline{Q}\|_{L^\infty} dt \leq 2\pi \int_T^{T+\delta} \|Q_{\sigma\sigma}\|_{L^2} dt \leq C_2 e^{-\frac{a}{2}T}.$$

Along 4- Q flow (1.2), $v(\theta, t)$ satisfies $v^{-1}v_t = -\frac{3}{4}(Q - \overline{Q})$. Integrating from T to $T + \delta$ we obtain that

$$|\ln v(\theta, T + \delta) - \ln v(\theta, T)| \leq C_3 e^{-\frac{a}{2}T},$$

for any $\theta \in [0, 2\pi]$, $T > 0$ and $\delta \in [0, 1]$. Hence $\lim_{t \rightarrow \infty} v(\theta, t) = v_\infty(\theta)$, with $\|v(t) - v_\infty\|_{L^\infty} \leq C_4 e^{-\frac{a}{2}t}$ and the 4- Q curvature of $g_\infty := v_\infty^{-\frac{4}{3}}g_s$ is constant 1. This completes the proof of Theorem 2.

5. CLASSIFICATION OF METRICS WITH CONSTANT 1- Q -CURVATURE

In this section we shall focus on proving Theorem 2.

Consider the functional

$$\mathbf{F}(u) = \frac{1}{9} \int_0^{2\pi} (u_{\theta\theta}^2 - 10u_\theta^2 + 9u^2) d\theta \left(\int_0^{2\pi} u^{-2/3}(\theta) d\theta \right)^3.$$

From the proof of Theorem 3 in [8] we know that

$$\inf_{u \in H^2(S^2), \text{ satisfying (1.3)}} F(u)$$

is achieved by $v \in H^2(S^2)$, which satisfies (1.3) and the Euler-Lagrange equation

$$(5.1) \quad v_{\theta\theta\theta\theta} + 10v_{\theta\theta} + 9v = \tau v^{-3}$$

for a positive constant τ .

Define $V : \mathbf{R} \rightarrow \mathbf{R}$ by

$$V(y) = v(\arctan(2y)) \left(\frac{1}{2} + 2y^2 \right)^{\frac{3}{2}}.$$

From conformal invariant properties of P_g (Proposition 1), we obtain that $V(y)$ satisfies

$$(5.2) \quad V''''(y) = \tau V(y)^{-\frac{5}{3}} \quad \text{in } \mathbf{R}.$$

Lemma 7. *Let $w(\theta) = v(\theta)^{\frac{1}{3}}$. Then w satisfies*

$$(5.3) \quad w^5(\theta)(w''(\theta) + w(\theta)) = w^5(\theta + \pi)(w''(\theta + \pi) + w(\theta + \pi)), \text{ for all } \theta \in [0, 2\pi].$$

Proof. The first integral of equation (5.2) is

$$V'V''' - \frac{1}{2}(V'')^2 = -\frac{3}{2}\tau V^{-\frac{2}{3}} + C.$$

Since $\lim_{y \rightarrow \pm\infty} V(y) = +\infty$, we have

$$\lim_{y \rightarrow \pm\infty} (V'V''' - \frac{1}{2}(V'')^2) = C.$$

Direct computation shows that for $y \rightarrow \pm\infty$,

$$V'V''' - \frac{1}{2}(V'')^2 = 18v^2(\pm\frac{\pi}{2}) - 4(v'(\pm\frac{\pi}{2}))^2 + 6v(\pm\frac{\pi}{2})v''(\pm\frac{\pi}{2}) + O(y^{-2}).$$

Since $w^5(w'' + w) = v^2 - \frac{2}{9}(v')^2 + \frac{1}{3}vv''$, we know that

$$C = 18v^2(\pm\frac{\pi}{2}) - 4(v'(\pm\frac{\pi}{2}))^2 + 6v(\pm\frac{\pi}{2})v''(\pm\frac{\pi}{2}) = 18w^5(\pm\frac{\pi}{2})(w''(\pm\frac{\pi}{2}) + w(\pm\frac{\pi}{2})).$$

By applying an arbitrary shift $\theta \rightarrow \theta + \beta$, we obtain (5.3). \square

Due to intermediate value theorem we may assume without loss of generality that $v(\frac{\pi}{2}) = v(-\frac{\pi}{2})$. The next lemma indicates that the main difficult in the proof of Theorem 2 is to match the derivative of v at north pole with that at south pole.

Lemma 8. *If we also have $v'(\frac{\pi}{2}) = v'(-\frac{\pi}{2})$, then $v^{(k)}(\frac{\pi}{2}) = v^{(k)}(-\frac{\pi}{2})$ for all $k \in \mathbf{N}$. Moreover*

$$v(\theta) = c(\lambda^2 \cos^2(\theta - \beta) + \lambda^{-2} \sin^2(\theta - \beta))^{3/2},$$

for some $\beta, \lambda, c > 0$.

Proof. Let $w = v^{\frac{1}{3}}$. Since $w' = \frac{1}{3}v^{-\frac{2}{3}}v'$, we have $w'(-\frac{\pi}{2}) = w'(\frac{\pi}{2})$. Using Lemma 7, we obtain that $w''(\frac{\pi}{2}) = w''(-\frac{\pi}{2})$. It follows from $v'' = 3w^2w'' + 6w(w')^2$ that $v''(\frac{\pi}{2}) = v''(-\frac{\pi}{2})$. Differentiating (5.3) and using induction, we obtain that $v^{(k)}(\frac{\pi}{2}) = v^{(k)}(-\frac{\pi}{2})$ for all $k \in \mathbf{N}$. It follows that $g(\theta) = v(\theta/2)$ is smooth on S^1 . Furthermore $g(\theta)$ satisfies

$$g'''' + \frac{5}{2}g'' + \frac{9}{16}g = \frac{\tau}{16}g^{-\frac{5}{3}}.$$

Using the same argument as in the last part of the proof of Theorem 4 in [8], we obtain that

$$g(\theta) = c\left(\lambda^2 \cos^2 \frac{\theta - \beta}{2} + \lambda^{-2} \sin^2 \frac{\theta - \beta}{2}\right)^{3/2},$$

for some $\beta, \lambda, c > 0$ and the Lemma follows. \square

The proof of Theorem 2 is thus completed if $v'(\frac{\pi}{2}) = v'(-\frac{\pi}{2})$. We are left to consider the case of $v'(\frac{\pi}{2}) \neq v'(-\frac{\pi}{2})$.

For any $a \in \mathbf{R}$, let $V_a(y) = V(y - a)$. Then $V_a(y)$ also satisfies (5.2). It follows that $\tilde{v}(\theta) := V_a(\frac{1}{2} \tan \theta)(2 \cos^2 \theta)^{\frac{3}{2}}$ satisfies (5.1) for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Observing that

$$\tilde{v}(\theta) = V_a(\frac{1}{2} \tan \theta)(2 \cos^2 \theta)^{\frac{3}{2}} = v(\arctan(\tan \theta - 2a))f(\theta),$$

where $f(\theta) := (\cos^2 \theta + (\sin \theta - 2a \cos \theta)^2)^{\frac{3}{2}}$, we obtain that (recall: $v(\pi/2) = v(-\pi/2)$)

$$v_a(\theta) := \begin{cases} v(\arctan(\tan \theta - 2a))f(\theta), & \text{when } \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\ v(\pi + \arctan(\tan \theta - 2a))f(\theta), & \text{when } \theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) \\ v(\frac{\pi}{2}), & \text{when } \theta = \pm\frac{\pi}{2} \end{cases}$$

is smooth on S^1 and satisfies (5.1) and (1.3) (see, for example, Section 6 in [8] for more details). Furthermore

$$v_a''(\pm\frac{\pi}{2}) = 24a^2v(\pm\frac{\pi}{2}) + 8av'(\pm\frac{\pi}{2}) + v''(\pm\frac{\pi}{2}).$$

Choosing (recall: $v'(\frac{\pi}{2}) \neq v'(-\frac{\pi}{2})$)

$$a = -\frac{1}{8} \frac{v''(\frac{\pi}{2}) - v''(-\frac{\pi}{2})}{v'(\frac{\pi}{2}) - v'(-\frac{\pi}{2})},$$

we have $v_a''(\frac{\pi}{2}) = v_a''(-\frac{\pi}{2})$. Let $w_a = v_a^{\frac{1}{3}}$. We have $w_a(\frac{\pi}{2}) = w_a(-\frac{\pi}{2})$, and thus $w_a''(\frac{\pi}{2}) = w_a''(-\frac{\pi}{2})$ by Lemma 7. From $v_a'' = 3w_a^2 w_a'' + 6w_a(w_a')^2$ we derive that $v_a'(\frac{\pi}{2}) = v_a'(-\frac{\pi}{2})$. It then follows from Lemma 8 that $v_a^{(k)}(\frac{\pi}{2}) = v_a^{(k)}(-\frac{\pi}{2})$ for all $k \in \mathbb{N}$ and

$$v_a(\theta) = c \left(\lambda^2 \cos^2(\theta - \beta) + \lambda^{-2} \sin^2(\theta - \beta) \right)^{3/2},$$

for some $\lambda, c > 0$ and $\beta \in [0, 2\pi)$. Direct computation shows that v has the form (1.4). This completes the proof of Theorem 2.

If v satisfies (5.1), then $g = v^{-\frac{4}{3}}g_0$ has constant 1- Q curvature. So we classify all the constant 1- Q curvature metrics on S^1 satisfying (1.3).

6. EXPONENTIAL CONVERGENCE OF THE 1- Q -FLOW

Based on the classification result in Theorem 2, we shall prove the exponential convergence of the 1- Q flow using a similar argument in Section 4.

Suppose that $g(t) = v^{-\frac{4}{3}}(\theta, t)g_s$ is a solution to the flow equation (1.2) for $\alpha = 1$ with initial metric satisfying (1.3). By Lemma 3, we know that the metric will satisfies (1.3) for all $t \geq 0$. As in the proof of Proposition 3, for any $t \in [0, \infty)$, choose $\lambda = \lambda(t) > 0, \beta = \beta(t) \in [0, 2\pi)$ so that $u(\theta, t) := (\mathbf{T}_{\lambda, \beta} v)(\theta)$ satisfies $\int_0^{2\pi} u(t) \cos 2\theta d\theta = \int_0^{2\pi} u(t) \sin 2\theta d\theta = 0$. Here $(\mathbf{T}_{\lambda, \beta} v)(\theta) = v(\sigma_{\lambda, \beta}(\theta))\Gamma_{\lambda, \beta}(\theta)$,

$$\Gamma_{\lambda, \beta}(\theta) = \left(\lambda^2 \cos^2(\theta - \beta) + \lambda^{-2} \sin^2(\theta - \beta) \right)^{\frac{3}{2}}$$

and $\sigma_{\lambda, \beta}(\theta) = \beta + \int_{\beta}^{\theta} \Gamma_{\lambda, \beta}^{-\frac{2}{3}} d\theta$.

As in Section 4, using the classification result (Theorem 2) we can prove that $\lim_{t \rightarrow \infty} u(\theta, t) = 1$ and $Q_{\infty} := \lim_{t \rightarrow \infty} Q = 1$.

Define variable γ as the inverse of θ under map $\sigma_{\lambda(t), \beta(t)}$, that is $\sigma_{\lambda, \beta}(\gamma) = \theta$. Then we have $d\sigma = v(\theta)^{-\frac{2}{3}} d\theta = u(\gamma)^{-\frac{2}{3}} d\gamma$. Since the 1-curvature and 1- Q curvature of metric $d\gamma \otimes d\gamma$ is 1, we obtain that

$$\begin{aligned} R(\theta(\gamma)) &= u(\gamma)((u^{\frac{1}{3}})''(\gamma) + u^{\frac{1}{3}}(\gamma)), \\ Q(\theta(\gamma)) &= u^{\frac{5}{3}}(\gamma) \left(\frac{1}{9} u_{\gamma\gamma\gamma}(\gamma) + \frac{10}{9} u_{\gamma\gamma}(\gamma) + u(\gamma) \right). \end{aligned}$$

It follows that $\lim_{t \rightarrow \infty} R(\theta) = 1$. Therefore

$$\begin{aligned} \partial_t G_2 &= -\frac{1}{6} \|Q_{\sigma\sigma}\|_{L^2}^2 + \frac{5}{3} \int_0^{2\pi} R Q_{\sigma}^2 d\sigma - 4\overline{Q} \|Q - \overline{Q}\|_{L^2}^2 - \frac{7}{2} \int_0^{2\pi} (Q - \overline{Q})^3 d\sigma \\ (6.1) \quad &= -\frac{1}{6} \|Q_{\sigma\sigma}\|_{L^2}^2 + \left(\frac{5}{3} + o(1) \right) \|Q_{\sigma}\|_{L^2}^2 - (4 - o(1)) \|Q - \overline{Q}\|_{L^2}^2, \end{aligned}$$

where $o(1) \rightarrow 0$ as $t \rightarrow +\infty$. Write

$$(6.2) \quad Q = \overline{Q} + \sum_{n=1}^{\infty} (a_n \cos(n\sigma) + b_n \sin(n\sigma)) = \tilde{c} + \sum_{n=1}^{\infty} (\tilde{a}_n \cos(n\gamma) + \tilde{b}_n \sin(n\gamma)).$$

Since $u(t) \rightarrow 1$ as $t \rightarrow \infty$, we obtain that

$$(6.3) \quad a_n = \tilde{a}_n + o(1)G_2^{\frac{1}{2}}, \quad b_n = \tilde{b}_n + o(1)G_2^{\frac{1}{2}}, \quad n = 0, 1, 2, 3, \dots,$$

where $a_0 = \overline{Q}$ and $\tilde{a}_0 = \tilde{c}$. We need the following lemma to estimate a_2, b_2 .

Lemma 9. *For ϕ smooth we have*

$$\int_0^{2\pi} (\phi^{(4)}(\theta) + 10\phi''(\theta) + 9\phi(\theta)) \left(\frac{1}{3}\phi'(\theta) \cos(2\theta) + \phi(\theta) \sin(2\theta) \right) d\theta = 0.$$

A special case of Lemma 9 is the following Kazdan-Warner type identity.

Corollary 2. *Given $g = v^{-\frac{4}{3}}g_s$ with v smooth. Then 1- Q curvature of g satisfies*

$$(6.4) \quad \int_0^{2\pi} Q_\theta v^{-\frac{2}{3}} \cos(2\theta) d\theta = \int_0^{2\pi} Q_\theta v^{-\frac{2}{3}} \sin(2\theta) d\theta = 0.$$

The proofs of Lemma 9 and Corollary 2 are very similar to the proofs of Lemma 6 and Corollary 1. We shall skip all the details here.

Using (6.4) and $\lim_{t \rightarrow \infty} u = 1$ we have

$$\begin{aligned} \tilde{a}_2 &= \frac{1}{\pi} \int_0^{2\pi} Q \cos(2\gamma) d\gamma = -\frac{1}{2\pi} \int_0^{2\pi} Q_\gamma \sin(2\gamma) d\gamma \\ &= -\frac{1}{2\pi} \int_0^{2\pi} Q_\gamma \sin(2\gamma) (u^{-\frac{2}{3}}(\gamma) - u^{-\frac{2}{3}}(\gamma) + 1) d\gamma \\ &= -\frac{1}{2\pi} \int_0^{2\pi} Q_\gamma \sin(2\gamma) (-u^{-\frac{2}{3}}(\gamma) + 1) d\gamma + 0 \\ &= o(1) \left(\int_0^{2\pi} Q_\gamma^2 d\gamma \right)^{\frac{1}{2}} = o(1) \|Q_\sigma\|_{L^2(d\sigma)}. \end{aligned}$$

Similarly $\tilde{b}_2 = o(1) \|Q_\sigma\|_{L^2}$. It follows from (6.3) that $a_2, b_2 = o(1) \|Q_\sigma\|_{L^2}$. From the Fourier expansion (6.2) of Q and (6.1) we obtain that

$$\partial_t G_2 = \pi \sum_{n \in \mathbb{N}, n \neq 2} \left(-\frac{1}{6}n^4 + \frac{5}{3}n^2 - 4 \right) (a_n^2 + b_n^2) + o(\|Q_\sigma\|_{L^2}).$$

Hence there exists $a > 0$, such that $\partial G_2 \leq -aG_2$, which implies that

$$G_2(t) \leq Ce^{-at}, \text{ for some } C > 0.$$

The rest of the proof can be carried out similarly to the proof of Theorem 1. We hereby complete the proof of Theorem 3.

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